

# CONTINUOUS $\times p, \times q$ -INVARIANT MEASURES ON THE UNIT CIRCLE

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ABSTRACT. We express continuous  $\times p, \times q$ -invariant measures on the unit circle via some simple forms. On one hand, a continuous  $\times p, \times q$ -invariant measure is the weak-\* limit of average of Dirac measures along an irrational orbit. On the other hand, a continuous  $\times p, \times q$ -invariant measure is a continuous function on  $[0, 1]$  satisfying certain function equations.

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## 1. INTRODUCTION

In [F76], H. Furstenberg shows that when  $\frac{\log p}{\log q}$  is irrational, every irrational orbit under  $\times p, \times q$  is dense in the unit circle  $\mathbb{T}$ . He also

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conjectures that the only continuous ergodic  $\times p, \times q$ -invariant measure is the Lebesgue measure.

In this paper, we express continuous  $\times p, \times q$ -invariant measures on the unit circle via two simple forms. One is an average of Dirac measures and the other one is homeomorphisms on  $[0, 1]$ .

The first says the following.

**Theorem 3.7.** *If  $\mu$  is an ergodic  $\times p, \times q$ -invariant continuous Borel probability measure on  $\mathbb{T}$ , then there exists an irrational  $x \in [0, 1)$  such that*

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \delta_{p^i q^j x} = \mu$$

under weak-\* topology, where  $\delta_y$  is the Dirac measure on  $[0, 1)$  concentrating at a point  $y \in [0, 1)$ .

The second is a conjecture equivalent to Furstenberg's conjecture.

**Conjecture 4.7.** *The only homeomorphism  $f$  on  $[0, 1]$  satisfying  $f = T_p f = T_q f$  is the identity. Here for a positive integer  $n$ , the operator  $T_n : C[0, 1] \rightarrow C[0, 1]$  is given by  $T_n g(x) = \sum_{i=0}^{n-1} g(\frac{x+i}{n}) - g(\frac{i}{n})$  for every  $x \in [0, 1]$  and  $g \in C[0, 1]$ .*

This can be taken as a real-value function version of [D11, Prop. 11].

## 2. PRELIMINARY

**2.1. Conventions.** Within this article, we denote the unit circle  $\{z \in \mathbb{C} \mid |z| = 1\}$  by  $\mathbb{T}$  (if necessary  $\mathbb{T}$  will be also denoted by  $\mathbb{R}/\mathbb{Z}$ ). Denote the set of nonnegative integers by  $\mathbb{N}$ , the set of positive integers by  $\mathbb{Z}^+$  and the function  $\exp 2\pi i x$  for  $x \in \mathbb{R}$  by  $e(x)$  and the function  $e(kx)$  by  $z^k$  for every  $k \in \mathbb{Z}$ . The notation  $C(X)$  stands for the set of continuous functions on a compact Hausdorff space  $X$ .

A measure always means a Borel probability measure. By identifying  $\mathbb{T}$  with  $[0, 1)$ , a measure on  $\mathbb{T}$  amounts to a measure on  $[0, 1)$ .

We call a number  $a \in \mathbb{T}$  rational if  $a = e(x)$  for some rational  $x \in [0, 1)$ , otherwise call  $a$  irrational. The greatest common divisor of  $m, n \in \mathbb{Z}^+$  is denoted by  $\gcd(m, n)$ .

Let  $\omega = \{x_n\}_{n=1}^\infty$  be a sequence of real numbers contained in the unit interval  $[0, 1)$  and for any positive integer  $N$  and a subset  $E \subseteq [0, 1)$ , denote  $\frac{|\{x_1, \dots, x_N\} \cap E|}{N}$  by  $A(E; N; \omega)$  or briefly  $A(E; N)$  if no confusion caused.

For a double sequence  $\omega = \{s_{ij}\}_{i,j=0}^\infty \subseteq [0, 1)$ , positive integers  $N, M$  and a subset  $E \subseteq [0, 1)$ , denote  $\frac{|\{s_{ij} | 0 \leq i \leq M-1, 0 \leq j \leq N-1\} \cap E|}{NM}$  by  $A(E; N, M; \omega)$  or briefly  $A(E; N, M)$ .

## 2.2. Equidistributed sequences in $\mathbb{T}$ .

**Definition 2.1.** [Equidistributed (double) sequences]

A sequence  $\{a_n\}_{n=1}^\infty$  in  $\mathbb{T}$  is called **equidistributed** if the sequence  $\omega = \{x_n\}_{n=1}^\infty$  in  $[0, 1)$  with  $e(x_n) = a_n$  satisfies

$$\lim_{N \rightarrow \infty} A([a, b]; N; \omega) = b - a,$$

for any  $0 \leq a < b \leq 1$ , or equivalently one can say the sequence  $\{x_n\}_{n=1}^\infty$  is uniformly distributed modulo 1 (u.d. mod 1) [KN74, Defn. 1.1].

A double sequence  $\{a_{i,j}\}_{i,j=0}^\infty$  in  $\mathbb{T}$  is called **equidistributed** if the sequence  $\omega = \{x_{ij}\}_{i,j=0}^\infty$  in  $[0, 1)$  such that  $e(x_{ij}) = a_{ij}$  satisfies

$$\lim_{N, M \rightarrow \infty} A([a, b]; N, M; \omega) = b - a,$$

for any  $0 \leq a < b \leq 1$ , or equivalently one can say the sequence  $\{x_{i,j}\}_{i,j=0}^\infty$  is uniformly distributed modulo 1 (u.d. mod 1) [KN74, Defn. 2.1].

For equidistributed sequences and equidistributed double sequences, one have corresponding Weyl's criterion [KN74, Thm. 2.1 & Thm. 2.9].

**Theorem 2.2.** [Weyl's criteria]

A (double) sequence  $\{a_n\}_{n=1}^\infty$  ( $\{a_{i,j}\}_{i,j=0}^\infty$ ) is equidistributed on  $\mathbb{T}$  iff

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N a_n^k &= 0 \\ \left( \lim_{N, M \rightarrow \infty} \frac{1}{NM} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} a_{ij}^k \right) &= 0, \end{aligned}$$

for every  $k \in \mathbb{Z}^+$ .

Equivalently one have the following

**Theorem 2.3.** [KN74, Thm. 1.1 & Thm. 2.8]

A (double) sequence  $\{a_n\}_{n=1}^\infty$  ( $\{a_{i,j}\}_{i,j=0}^\infty$ ) in  $\mathbb{T}$  is equidistributed iff

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(a_n) &= \int_{\mathbb{T}} f(z) dm(z) \\ \left( \lim_{N, M \rightarrow \infty} \frac{1}{NM} \sum_{i=0}^{N-1} \sum_{j=0}^{M-1} f(a_{ij}) \right) &= \int_{\mathbb{T}} f(z) dm(z), \end{aligned}$$

for every  $f \in C(\mathbb{T})$ . Here  $m$  is the Lebesgue measure of  $\mathbb{T}$ .

A weaker version of equidistribution of double sequences is the following [KN74, The paragraph before Lemma 2.4].

**Definition 2.4.** A double sequence  $\{a_{i,j}\}_{i,j=0}^\infty \subset \mathbb{T}$  is called **equidistributed in the squares** on  $\mathbb{T}$  if the sequence  $\omega = \{x_{ij}\}_{i,j=0}^\infty$  in  $[0, 1)$  with  $e(x_{ij}) = a_{ij}$  satisfies

$$\lim_{N \rightarrow \infty} A([a, b]; N, N; \omega) = b - a,$$

for any  $0 \leq a < b \leq 1$ .

Similarly  $\{a_{i,j}\}_{i,j=0}^\infty \subset \mathbb{T}$  is equidistributed in the squares on  $\mathbb{T}$  iff

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} a_{ij}^k = 0$$

for every  $k \in \mathbb{Z}^+$ .

### 3. EQUIDISTRIBUTED DOUBLE SEQUENCES AND ERGODIC $\times p, \times q$ INVARIANT MEASURES

**3.1. Equidistributed irrational orbits.** From now on, we fix two positive integers  $p, q$  such that  $\frac{\log p}{\log q} \notin \mathbb{Q}$  (the multiplicative semigroup  $\{p^i q^j\}_{i,j \in \mathbb{N}} \not\subseteq \{a^n\}_{n \in \mathbb{N}}$  for every  $a \in \mathbb{Z}^+$ ).

In this section, we show that every ergodic  $\times p, \times q$ -invariant measure  $\mu$  on  $\mathbb{T}$  can be written as the weak-\* limit of  $\{\frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \mu_{a^{p^i q^j}}\}_{N=1}^\infty$  for some  $a \in \mathbb{T}$ .

**Definition 3.1.** [Generic point]

A point  $a \in \mathbb{T}$  is called **generic** with respect to an ergodic  $\times p, \times q$ -invariant measure  $\mu$  of  $\mathbb{T}$  if

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(a^{p^i q^j}) = \mu(f)$$

for all  $f \in C(\mathbb{T})$ . Denote the set of generic points with respect to  $\mu$  by  $X_\mu$ .

**Definition 3.2.** [Amenable semigroup] [B71, p. 2] [OW83, p. 2]

A countable discrete semigroup  $P$  is called **amenable** if there exists a sequence  $\{F_n\}_{n=1}^\infty$  of finite subsets of  $P$  such that

$$\lim_{n \rightarrow \infty} \frac{|sF_n \triangle F_n|}{|F_n|} = 0$$

for any  $s \in P$ , and  $\{F_n\}_{n=1}^\infty$  is called a (left) Følner sequence. A Følner sequence  $\{F_n\}_{n=1}^\infty$  is called **special** if

- (1)  $F_n \subseteq F_{n+1}$ ;
- (2) There exists some constant  $M > 0$  such that  $|F_n^{-1}F_n| \leq M|F_n|$  for all  $n \in \mathbb{Z}^+$ , where  $F_n^{-1}F_n = \{s \in P \mid ts \in F_n \text{ for some } t \in F_n\}$ .

Before proceeding to prove the main result, we need a pointwise ergodic theorem as a preliminary, which is a special case of [B71, Thm. 3].

**Theorem 3.3.** [Generalized Birkhoff pointwise ergodic theorem]

Suppose  $P$  is a discrete amenable semigroup and  $X$  is a compact Hausdorff space. Assume that there is a continuous, measure-preserving action of  $P$  on a Borel probability space  $(X, \mathcal{B}, \mu)$ , and  $\mu$  is an ergodic  $P$ -invariant measure. If  $P$  has a special Følner sequence  $\{F_n\}_{n=1}^\infty$ , then for every  $f \in L^1(X, \mu)$ , the sequence  $\{\frac{1}{|F_n|} \sum_{s \in F_n} f(s \cdot x)\}_{n=1}^\infty$  converges almost everywhere to a  $P$ -invariant function  $f^* \in L^1(X, \mu)$  such that  $\int_X f d\mu = \int_X f^* d\mu$ .

Using Theorem 3.3, we prove the following theorem which shows generic points with respect to an ergodic  $\times p, \times q$ -invariant measure  $\mu$  are almost everywhere.

**Theorem 3.4.** For every ergodic  $\times p, \times q$ -invariant measure  $\mu$  on  $\mathbb{T}$ , we have  $\mu(X_\mu) = 1$ .

*Proof.* Consider the measure preserving action of  $\mathbb{N}^2$  on  $(\mathbb{T}, \mu)$  given by  $\times p, \times q$ . Note that  $\mathbb{N}^2$  is an amenable semigroup with a special Følner sequence  $\{F_n\}_{n=1}^\infty$  given by  $F_n = \{(i, j) \mid 0 \leq i, j \leq n-1\}$ . Since  $\mu$  is ergodic, every  $\mathbb{N}^2$ -invariant function in  $L^1(\mathbb{T}, \mu)$  is constant. Applying Theorem 3.3, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} S^i T^j(f)(x) = \mu(f)$$

for every  $f \in C(\mathbb{T})$  and almost every  $x \in \mathbb{T}$  with respect to  $\mu$ . Denote the set of such points for  $f$  by  $X_f$ . Then  $\mu(X_f) = 1$ .

Take a countable dense set  $\{f_n\}_{n=1}^\infty$  in  $C(\mathbb{T})$ . Then it is easy to see that  $X_\mu = \bigcap_{n=1}^\infty X_{f_n}$  and hence  $\mu(X_\mu) = 1$ .  $\square$

**Corollary 3.5.** If  $\mu$  is finitely supported, then  $\text{Supp}(\mu)$ , the support of  $\mu$  is a subset of  $X_\mu$ .

*Proof.* Since  $\mu$  is atomic, the set  $\text{Supp}(\mu)$  consists of finitely many atoms. Hence every atom is in  $X_\mu$  otherwise  $\mu(X_\mu) < 1$ .  $\square$

Next we prove that every rational is a generic point with respect to an atomic ergodic  $\times p, \times q$ -invariant measure.

**Lemma 3.6.** If  $x, y \in [0, 1)$  are in the same orbit under  $\times p, \times q$  (which means  $x = p^i q^j y \pmod{1}$  for some  $i, j \in \mathbb{Z}$ , then  $x \in X_\mu$  iff  $y \in X_\mu$ .

*Proof.* Let  $a = e(x)$  and  $b = e(y)$ . There exists  $c \in \mathbb{T}$  such that  $c = a^{p^m q^n} = b^{p^k q^l}$  for some  $k, l, m, n \in \mathbb{N}$ . The proof follows from

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(a^{p^i q^j}) &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(c^{p^i q^j}) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} f(b^{p^i q^j}) \end{aligned}$$

(if any of these three limits exists) for all  $f \in C(\mathbb{T})$ .  $\square$

A finite Borel measure  $\mu$  on  $\mathbb{T}$  is called **continuous** or **non-atomic** if  $\mu\{z\} = 0$  for every  $z \in \mathbb{T}$ .

**Theorem 3.7.** Every rational  $a \in \mathbb{T}$  is a generic point with respect to a finitely supported ergodic  $\times p, \times q$ -invariant measure. Hence for an ergodic  $\times p, \times q$ -invariant continuous measure  $\mu$  on  $\mathbb{T}$ , there exists an irrational  $x \in [0, 1)$  such that

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \delta_{p^i q^j x} = \mu$$

under weak-\* topology.

*Proof.* Let  $a = e(\frac{m}{n})$  for  $m, n \in \mathbb{Z}^+$  with  $\gcd(m, n) = 1$ . Then there exist  $s, t \in \mathbb{Z}^+$  such that

- $\frac{m}{n}$  and  $\frac{s}{t}$  are in the same orbit under  $\times p, \times q$ .
- $\gcd(s, t) = 1$  and  $\gcd(t, pq) = 1$ .

There exists an ergodic  $\times p, \times q$ -invariant measure  $\mu$  finitely supported in  $\{\frac{j}{s} | 0 \leq j \leq s-1\}$  such that  $\frac{s}{t}$  is in  $\text{Supp}(\mu)$ . Combining Corollary 3.5 and Lemma 3.6, we finish the proof of the first part. The second part follows immediately.  $\square$

#### 4. INVARIANT SUBSPACE OF $C[0, 1]$ UNDER AN ACTION OF A MULTIPLICATIVE SEMIGROUP $\Sigma$ OF $\mathbb{N}$ .

##### 4.1. $\times p, \times q$ -invariant measures via continuous functions on $[0, 1]$ .

**Definition 4.1.** For a positive integer  $n \geq 2$ , define  $T_n : C[0, 1] \rightarrow C[0, 1]$  by

$$T_n(g)(x) = \sum_{j=0}^{n-1} \left[ g\left(\frac{x+j}{n}\right) - g\left(\frac{j}{n}\right) \right]$$

for all  $g \in C[0, 1]$ . We say an  $f \in C[0, 1]$  is  $T_n$ -**invariant** if  $T_n(f) = f$ .

We see that  $T_n$  is a bounded linear operator under the norm  $\|g\| := \max_{x \in [0, 1]} |g(x)|$  for all  $g \in C[0, 1]$ .

**Lemma 4.2.** For  $n \geq 2$ , if  $f$  is  $T_n$ -invariant, then

$$\int_0^1 f(x) dx = \frac{\sum_{j=0}^{n-1} f(\frac{j}{n})}{n-1}. \quad (4.1)$$

*Proof.* Take integral from 0 to 1 on both sides, we get

$$\begin{aligned} \int_0^1 f(x) dx &= \int_0^1 \left[ \sum_{j=0}^{n-1} \left[ f\left(\frac{x+j}{n}\right) - f\left(\frac{j}{n}\right) \right] \right] dx \\ &= \sum_{j=0}^{n-1} n \int_{\frac{j}{n}}^{\frac{j+1}{n}} f(x) dx - \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right) \\ &= n \int_0^1 f(x) dx - \sum_{j=0}^{n-1} f\left(\frac{j}{n}\right). \end{aligned}$$

Then Equation 4.1 follows immediately.  $\square$

**Proposition 4.3.** For two positive integers  $n$  and  $m$ , we have  $T_n T_m = T_{nm}$ .

*Proof.*

$$\begin{aligned} T_n T_m f(x) &= \sum_{j=0}^{n-1} \left[ T_m f\left(\frac{x+j}{n}\right) - T_m f\left(\frac{j}{n}\right) \right] \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} \{ [f(\frac{\frac{x+j}{n} + k}{m}) - f(\frac{k}{m})] - [f(\frac{\frac{j}{n} + k}{m}) - f(\frac{k}{m})] \} \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} [f(\frac{x+j+nk}{mn}) - f(\frac{j+nk}{mn})] \\ &= \sum_{j=0}^{mn-1} [f(\frac{x+j}{mn}) - f(\frac{j}{mn})] = T_{nm} f(x). \end{aligned}$$

$\square$

Let  $\Sigma$  be a multiplicative semigroup of  $\mathbb{N}$ . By Proposition 4.3, we have a semigroup action of  $\Sigma$  on  $C[0, 1]$  given by  $T_n$  for all  $n \in \Sigma$ . We say an  $f \in C[0, 1]$  is  $\Sigma$ -invariant if  $f$  is  $T_n$ -invariant for all  $n \in \Sigma$ .

Next we show some connection between continuous  $\Sigma$ -invariant measures on  $\mathbb{T}$  and  $\Sigma$ -invariant functions in  $C[0, 1]$ .

Given a probability measure  $\mu$  on  $\mathbb{T}$ , identify  $\mathbb{T}$  with  $[0, 1)$ . Then  $\mu$  can be taken as a probability measure on  $[0, 1)$ , and if  $\mu$  is continuous, then  $D_\mu$  is a nondecreasing continuous function on  $[0, 1]$  and  $D_\mu(0) = 0, D_\mu(1) = 1$ .

**Definition 4.4.** For a probability measure  $\mu$  on  $[0, 1)$ , define the **distribution function** of  $\mu$ , denoted by  $D_\mu$ , by  $D_\mu(x) = \mu[0, x)$  for all  $x \in [0, 1]$ .

**Proposition 4.5.** Suppose  $\mu$  is a continuous measure on  $\mathbb{T}$ . Then  $\mu$  is  $\times n$ -invariant iff  $D_\mu$  is  $T_n$ -invariant.

*Proof.* Suppose  $\mu$  is  $\times n$ -invariant. For any  $x \in [0, 1]$ , the preimage of  $[0, x)$  under  $\times n$  is  $\bigcup_{i=0}^{n-1} [\frac{i}{n}, \frac{x+i}{n})$ . So we get

$$D_\mu(x) = \mu[0, x) = \sum_{i=0}^{n-1} \mu[\frac{i}{n}, \frac{x+i}{n}) = \sum_{i=0}^{n-1} D_\mu(\frac{x+i}{n}) - D_\mu(\frac{i}{n}) = T_n D_\mu(x).$$

On the other hand, assume that  $D_\mu$  is  $T_n$ -invariant. To show that  $\mu$  is  $\times n$ -invariant, we only need to check that  $\mu(z^k) = \mu(z^{kn})$  for all positive integers  $k$ . Here  $z = e^{2\pi i x}$ .

$$\begin{aligned} \mu(z^k) &= \int_0^1 e^{2\pi i k x} dD_\mu(x) = e^{2\pi i k x} D_\mu(x)|_0^1 - \int_0^1 D_\mu(x) d e^{2\pi i k x} \\ &= 1 - 2\pi i k \int_0^1 [\sum_{j=0}^{n-1} D_\mu(\frac{x+j}{n}) - D_\mu(\frac{j}{n})] e^{2\pi i k x} dx \\ &= 1 - 2\pi i k \int_0^1 [\sum_{j=0}^{n-1} D_\mu(\frac{x+j}{n})] e^{2\pi i k x} dx \\ &= 1 - 2\pi i k n \sum_{j=0}^{n-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} D_\mu(x) e^{2\pi i k (nx-j)} dx \\ &= 1 - 2\pi i k n \int_0^1 D_\mu(x) e^{2\pi i k n x} dx = \mu(z^{kn}). \end{aligned}$$

□

For a semigroup  $\Sigma \subseteq \mathbb{N}$ , denote the space of  $\Sigma$ -invariant functions by  $C_\Sigma[0, 1]$ .

**Theorem 4.6.** The Lebesgue measure is the only continuous  $\Sigma$ -invariant measure on  $\mathbb{T}$  if  $\dim C_\Sigma[0, 1] = 1$ .

*Proof.* Note that  $T_n x = x$  for all positive integer  $n$ . Hence  $\dim C_\Sigma[0, 1] \geq 1$ . If  $\dim C_\Sigma[0, 1] = 1$ , then  $C_\Sigma[0, 1]$  consists of functions of the form  $\alpha x$  for some complex number  $\alpha$ . Suppose  $\mu$  is a continuous  $\Sigma$ -invariant



measure on  $\mathbb{T}$ . By Proposition 4.5, the distribution function  $D_\mu$  is in  $C_\Sigma[0, 1]$  and  $D_\mu(1) = 1$ . So  $D_\mu(x) = x$ , which means that  $\mu$  is the Lebesgue measure.  $\square$

Consequently, if the following conjecture is true, then Furstenberg's conjecture is true.

**Conjecture 4.7.** The only  $f \in C[0, 1]$  satisfying that

- (1)  $f$  is non-decreasing (even by Furstenberg's classification result of closed  $\times p, \times q$ -invariant subsets of  $\mathbb{T}$ , we can assume that  $f$  is strictly increasing, hence a homeomorphism on  $[0, 1]$  with  $f(0) = 0$ );
- (2)  $f(x) = \sum_{i=0}^{p-1} f(\frac{x+i}{p}) - f(\frac{i}{p}) = \sum_{i=0}^{q-1} f(\frac{x+i}{q}) - f(\frac{i}{q})$ ,

is  $x$ .

**4.2. The Cantor function as a  $T_3$ -invariant function.** Although Furstenberg's conjecture is equivalent to a conjecture in the framework of calculus, the difficulty doesn't reduce at all. To get a feeling of this, we look at a concrete example, the Cantor function, which is  $T_3$ -invariant, but not a homeomorphism on  $[0, 1]$ .

**Definition 4.8.** [The Cantor function]

The **Cantor function**  $c : [0, 1] \rightarrow [0, 1]$  is defined via the following procedures:

- (1) Express  $x \in [0, 1]$  in base 3;
- (2) If  $x$  contains a 1, replace every digit after the first 1 by 0;
- (3) Replace all 2s with 1s;
- (4) Interpret the result as a binary number.

The result is  $c(x)$ .

Notice that  $c(\frac{1}{3}) = c(\frac{2}{3}) = \frac{1}{2}$ , so  $c(x)$  is not a homeomorphism although it is a non-decreasing map from  $[0, 1]$  onto  $[0, 1]$ .

Let  $m(x) = \min\{n | x_n = 1\}$  for  $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} \in [0, 1]$  with  $0 \leq x_n \leq 2$ . When there is no  $n$  such that  $x_n = 1$ , let  $m(x) = \infty$ . Hence

$$c(x) = \sum_{n < m(x)} \frac{\frac{x_n}{2}}{2^n} + \frac{1}{2^{m(x)}} \quad (4.2)$$

for all  $x \in [0, 1]$ . If  $m(x) = 1$ , then  $c(x) = \frac{1}{2}$ . Of course, Equation 4.2 is nothing new, but in some sense, it is more explicit (hence more convenient) for us to prove some properties of  $c(x)$ .

Using Equation 4.2, the proof of the following lemma is straightforward.

**Lemma 4.9.** [Properties of  $c(x)$ ]

For every  $x \in [0, 1]$ , we have

- (1)  $c(\frac{x}{3}) = \frac{1}{2}c(x)$ ;
- (2)  $c(\frac{x+1}{3}) = \frac{1}{2}$ ;
- (3)  $c(\frac{x+2}{3}) = c(\frac{2}{3}) + c(\frac{x}{3}) = \frac{1}{2} + \frac{1}{2}c(x)$ .

Lemma 4.9 gives the following.

**Proposition 4.10.** The Cantor function  $c(x)$  is  $T_3$ -invariant.

*Proof.* By definition  $T_3c(x) = [c(\frac{x}{3}) + c(\frac{x+1}{3}) + c(\frac{x+2}{3})] - [c(0) + c(\frac{1}{3}) + c(\frac{2}{3})]$  for all  $x \in [0, 1]$ . It follows from Lemma 4.9 that

$$T_3c(x) = [\frac{c(x)}{2} + \frac{1}{2} + \frac{1}{2} + \frac{c(x)}{2}] - [0 + \frac{1}{2} + \frac{1}{2}] = c(x)$$

for all  $x \in [0, 1]$ . □

**Lemma 4.11.**  $c(x)$  is not a Lipschitz function.

*Proof.* Take  $x = 1$  and  $y$  such that  $y_n = 2$  for  $n < N$ ,  $y_N = 1$  and  $y_n = 0$  for  $n > N$ . Then  $|x - y| = \frac{2}{3^N}$  and  $c(x) - c(y) = 1 - (\frac{1}{2} + \dots + \frac{1}{2^N}) = \frac{1}{2^N}$ . Hence

$$\frac{|c(x) - c(y)|}{|x - y|} = \frac{1}{2} \left(\frac{3}{2}\right)^N.$$

As  $N \rightarrow \infty$ , we have  $y \rightarrow x$ , but  $\frac{|c(x) - c(y)|}{|x - y|} \rightarrow \infty$ . □

By Equation 4.1, we have

$$\int_0^1 c(x) dx = \frac{c(\frac{1}{3}) + c(\frac{2}{3})}{2} = \frac{1}{2}.$$

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